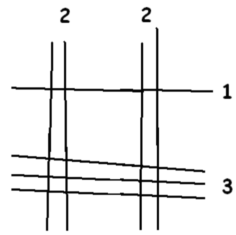
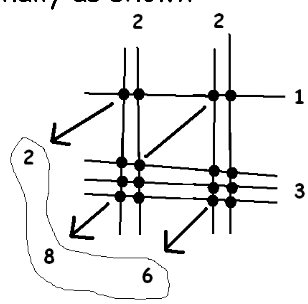
WEIRD MULTIPLICATION

Here's an unusual means for performing long multiplication. To compute 22×13, for example, draw two sets of vertical lines, the left set containing two lines and the right set two lines (for the digits in 22) and two sets of horizontal lines, the upper set containing one line and the lower set three (for the digits in 13).

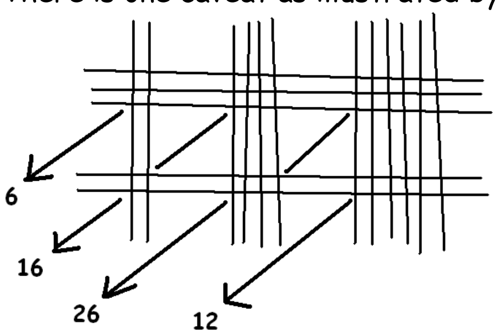


There are four sets of intersection points. Count the number of intersections in each and add the results diagonally as shown:



The answer 286 appears.

There is one caveat as illustrated by the computation 246×32:



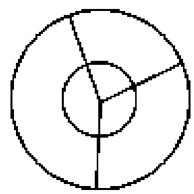
Although the answer 6 thousands, 16 hundreds, 26 tens, and 12 ones is absolutely correct, one needs to carry digits and translate this into 7,872.

a) Compute 131×122 via this method.
b) Compute 54×1332 via this method.
c) How best should one compute 102×30054 via this method?
d) Why does the method work?

A TINY TIDBIT:
Putting on one's shoes and then one's socks does not produce the same result as first putting on one's socks and then one's shoes. These two operations are not commutative.

In mathematics, multiplication is said to be commutative, meaning that $a \times b$ produces the same result as $b \times a$ for all numbers *a* and *b*. Is this obviously true? It might not be given the following graphical model for multiplication:

To compute 2×3, for instance, first draw two concentric circles to model the first number in the problem, 2, and three radii to model the second number, 3.



The number of separate pieces one sees in the diagram - in this case 6 - is the answer to the multiplication problem.

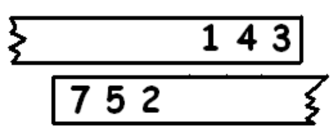
Is it at all obvious that drawing three concentric circles and two radii will produce the same number of pieces?

Suppose one draws 77 circles and 43 radii. Can we be sure that drawing 43 circles and 77 radii yields an equivalent number of pieces?

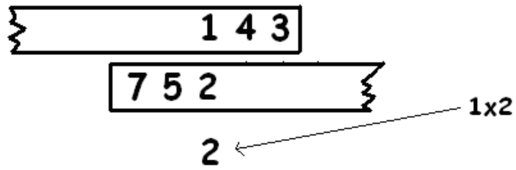
Is the commutativity of multiplication a belief or a fact?

ANOTHER TIDBIT:
Here's another way to compute a multiplication problem; 341×752, for instance.

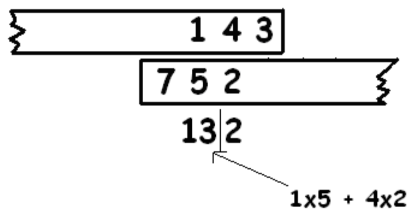
Write each number on a strip of paper, reversing one of the numbers. Start with the reversed number on the top strip to the right of the second number.



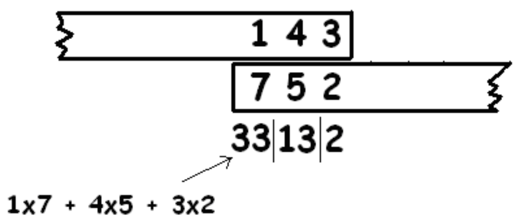
Slide the top number to the left until a first set of digits align. Multiply them and write their product underneath.

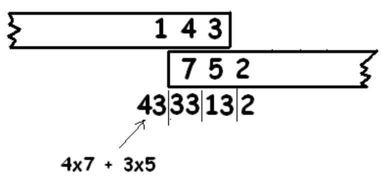


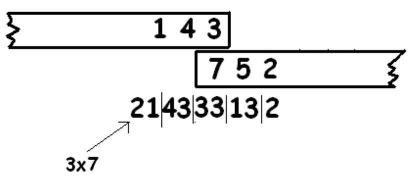
Slide one more place to the left and multiply the digits that are aligned. Write their sum underneath.



Continue this process:





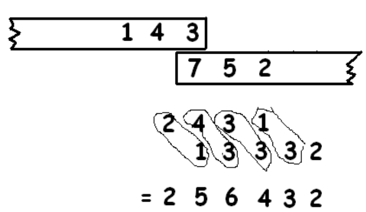


The answer to 341×752 is:

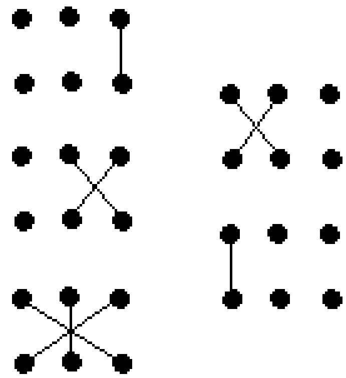
21 | 43 | 33 | 13 | 2

namely, 21 ten-thousands, 43 thousands, 33 hundreds, 13 tens and 2 ones. After carrying digits this translates to the number 256,432.

COMMENT: Some people may prefer to carry digits as they conduct this procedure and write something akin to:



ANOTHER PUZLER: Vedic mathematics taught in India (and established in 1911 by Jagadguru Swami Bharati Krishna Tirthaji Maharaj) has students compute the multiplication of two three-digit numbers as follows:



What do you think this sequence of diagrams means?

AND MORE!

FINGER MULTIPLICATION

Don't memorize your multiplication tables. Let your fingers do the work! If you are comfortable with multiples of two, three, four, and five, then there is an easy way to compute product values in the six- through ten times tables. First encode numbers this way:

A closed fist represents "five" and any finger raised on that hand adds "one" to that value.

Thus a hand with two fingers raised, for example, represents "seven" and a hand with three fingers raised represents "eight." To multiply two numbers between five and ten, do the following:

1. Encode the two numbers, one on each hand, and count "ten" for each finger raised.

2. Count the number of unraised *fingers on each hand and multiply together the two counts.*

3. Add the results of steps one and two. This is the desired product.

For example, "seven times eight" is represented as two raised fingers on the left hand, three on the right hand. There are five raised fingers in all, yielding the number "50" for step one. The left hand has three lowered fingers and the right, two. We compute: $3 \times 2 = 6$. Thus the desired product is $50 + 6 = 56$.

Similarly, "nine times seven" is computed as $60+1\times3=63$, and "nine times nine" as $80+1\times1=81$. Notice that one is never required to multiply two numbers greater than five!

<u>FINGERS AND TOES</u>:
One can compute higher products using the same method! For example, with fingers and toes, one interprets 17×18 as "seven raised fingers" and "eight raised toes." This time we count each raised digit as "twenty" (we have twenty digits fingers and toes in all!) yielding: $17\times18=20\times15+3\times2=306$!

QUESTION: Martians have six fingers on each of two hands. Describe their version of the finger multiplication trick.

RUSSIAN MULTIPLICATION

The following unusual method of multiplication believed to have originated in Russia.

1. Head two columns with the numbers you wish to multiply.

2. Progressively halve the numbers in the left column (ignoring remainders) while doubling the figures in the right column. Reduce the left column to one.

3. Delete all rows with an even number in the left and add all the numbers that survive in the right. This sum is the desired product!

37	x	23
~~18~~		~~46~~
9		92
~~4~~		~~184~~
~~2~~		~~368~~
1		736
		851

37 x 23 = 851

See also *THINKING MATHEMATICS! Volume 1* for details.

COMMENTARY and THOUGHTS

I have found the content of this newsletter to be a big hit with folks teaching middle-school students and, in general, a delight for students of all ages to ponder upon. Along with "finger multiplication" and "Russian multiplication" presented below, these techniques truly demonstrate the joy to be had engaging in intellectual play.

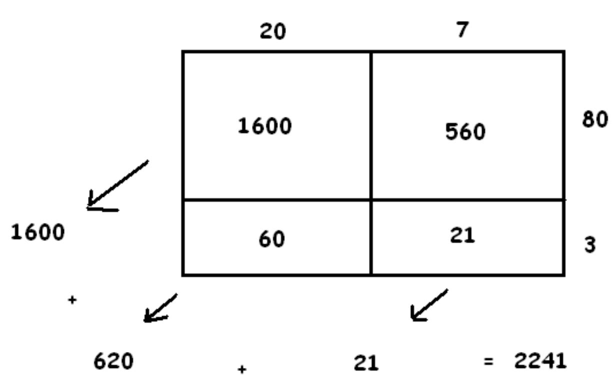
The multiplication methods outlined invite us to reexamine the long-multiplication algorithm we teach our young students. Consider, for example, the computation 83×27. Students are taught to write something of the ilk:

```
   8 3
 x 2 7
 -----
   2 1
  56 0
   6 0
 160 0
 -----
 22 4 1
```

Mathematicians recognize this as an exercise in "expanding brackets:"

$$83\times 27=(80+3)(20+7)=1600+60+560+21$$

Young students might be taught to associate an "area model" with multiplication, in which case, expanding brackets is equivalent to dividing a rectangle into computationally simpler pieces.

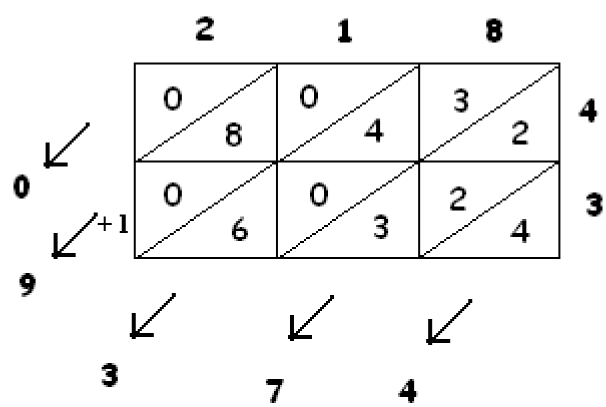


Notice that adding the numbers in the cells according to the diagonals on which they lie conveniently groups the 100s, the 10s and the units.

COMMENT: When I teach long multiplication to students, this area model is the only approach I adopt. I personally feel that the level of understanding that comes of it far outweighs the value of speed that comes from the standard algorithm: students remain acutely aware that the "8" in the above problem, for instance, means eighty and there is no mystery about placement of zeros and carrying digits.

ASIDE: In the 1500s in England students were taught to compute long multiplication using following *galley method* (also known as the *lattice method* or the *Elizabethan method*).

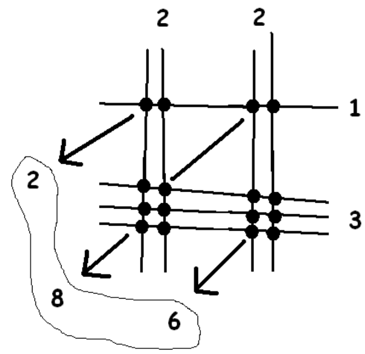
> *To multiply 218 and 43, for example, draw a* 2×3 *grid of squares. Write the digits of the first number along the top of the grid and the digits of the second number along the right side. Divide each cell of the grid diagonally and write in the product of the column digit and row digit of that cell, separating the tens from the units across the diagonal of that cell. (If the product is a one digit answer, place a 0 in the tens place.)*



> *Add the entries in each diagonal, carrying tens digits over to the next diagonal if necessary, to see the final answer. In our example, we have* $218\times 43 = 9374$.

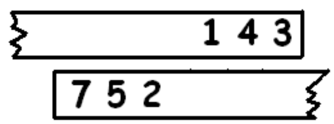
The diagonals serve the purpose of grouping together like powers of ten. (Although, again, the lack of zeros can make this process quite mysterious to young students.)

It is now easy to see that the lines method is just a area model of multiplication in disguise (since the count of intersection points matches the multiplication of the single digit numbers).



$22 \times 13 = 286$

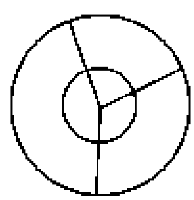
The strip method is a clever geometric tracking of the individual multiplications that occur in a long multiplication problem (with place-value keeping track of the powers of ten for us):



And the Vedic technique is a memorization device for the standard algorithm.

Circle/Radius Math

It is not hard to see that drawing a common radii for b circles yields ab pieces. (One radius produces b pieces and each radius inserted thereafter adds an additional b segments.)



Thus, "circle/radius math" is ordinary multiplication in disguise. (Is it? What happens if one or both of a or b is zero?) As ordinary multiplication is taken to be commutative, this multiplication must be commutative too. Without thinking through the arithmetic, this can be a surprise.

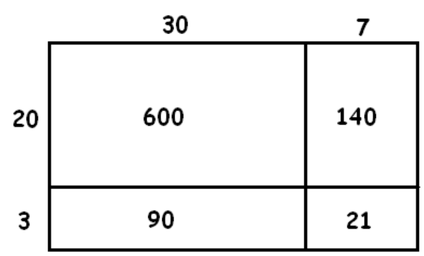
PEDAGOGICAL COMMENT: The Role of the Area Model for Multiplication

Mathematicians recognize that the area model for multiplication has its theoretical limitations—what is the area of a $-\sqrt{2}$ by $3+2i$ rectangle, for example?—but it does justify the axioms we care to work with when defining a ring. Pedagogically, it certainly seems to be an appropriate and intuitively accurate place for beginning understanding and investigation.

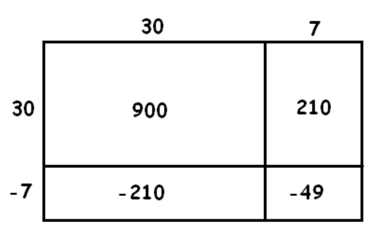
For example, if we allow for negative numbers in our considerations and consider rectangles with negative side-lengths (young students absolutely delight in "breaking the rules" in this way), then we can help justify many confusing features of the arithmetic we choose to believe. My favorite is tackling the issue of why the negative times negative should be positive.

In the world of counting numbers, it seems natural to say that 7×3 represents "seven groups of three" and so equals 21. In the same way, $7 \times (-3)$ represents "seven groups of negative three" and equals -21. It is difficult to interpret $(-7) \times 3$ in this way, but if we choose to extend the commutativity law to this realm, then we can say that this is the same as "three groups of negative seven" and so again is -21. Up to this point, I find that students (and teachers!) feel at ease and willing to say that all is fine. The kicker comes in trying to properly understand $(-7) \times (-3)$.

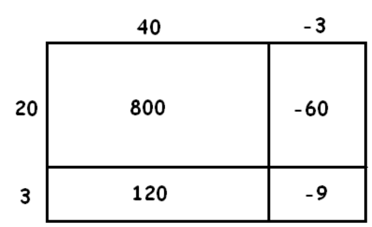
Let's do this by computing 23×17 multiple ways. Here are three:



23 x 37 = 600 + 90 + 140 + 21 = 851

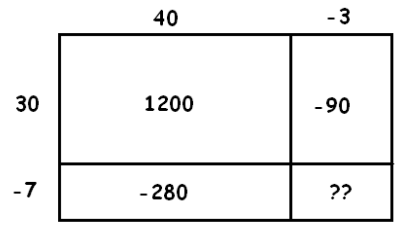


23 x 37 = 900 - 210 + 210 - 49 = 851



23 x 37 = 800 + 120 - 60 - 9 = 851

And this fourth, being the same computation, must yield the same answer:



23 x 37 = 1200 - 280 - 90 + ?? = 851

We thus have no choice but to set $(-7)\times(-3)$ equal to $+21$.(How do the axioms of a ring ensure that negative times negative must by positive for $\mathbb{Z}$? Where are those axioms in play in the above "area model" argument?)