CURRICULUM INSPIRATIONS: www.maa.org/ci
Math for America DC: www.mathforamerica.org/DC

Innovative Online Courses: www.gdaymath.com
Tanton Tidbits: www.jamestanton.com

WOW! COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

OCTOBER 2014

PROMOTIONAL CORNER: *Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.*

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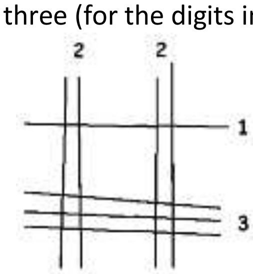
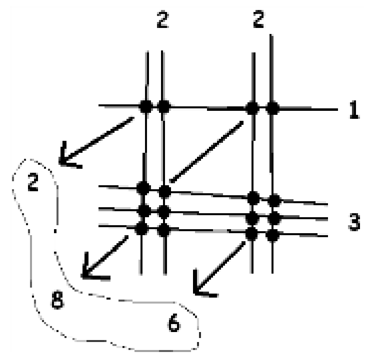
The MAA's Curriculum Inspirations project carries on with gusto and mighty good impact: www.maa.org/ci. Check out how to teach problem-solving, how to connect specific mathematical content with daily classroom practice, the curriculum, and the Common Core, and how to view any single mathematical question as a portal to more exploration, intrigue, joy, and mathematical delight!

LINE MULTIPLICATION PUZZLE

Here's an unusual way to perform long multiplication. To compute 22×13, for example, draw two sets of vertical lines, the left set containing two lines and the right set two lines (for the digits in 22) and two sets of horizontal lines, the upper set containing one line and the lower set three (for the digits in 13).

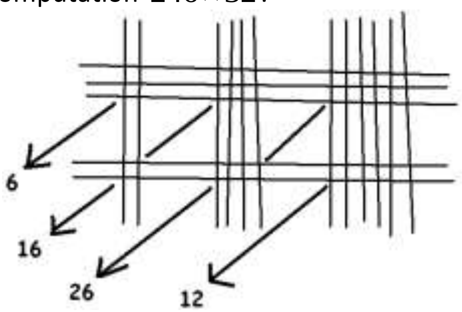


There are four sets of intersection points. Count the number of intersections in each group and add the results diagonally as shown:



The answer 286 appears.

There is one caveat as illustrated by the computation 246×32:



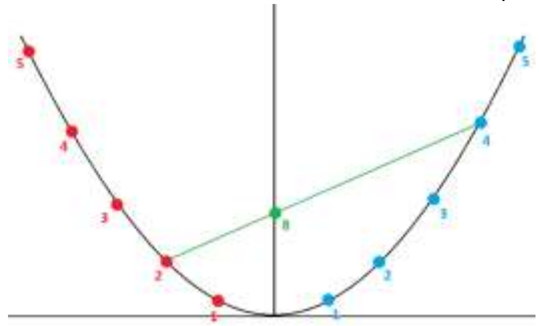
Although the answer 6 thousands, 16 hundreds, 26 tens, and 12 ones is absolutely correct, one needs to carry digits to translate this as 7,872.

a) Compute 131×122 via this method.
b) Compute 54×1332 via this method.
c) How best should one compute 102×30054 via this method?
d) Why does the method work?

See *Math Galore!* published by the MAA (2012) for this approach and additional weird approaches to multiplication.

PARABOLIC MULTIPLICATION

Here's a clever way to use a parabola to compute the product of two numbers. Draw the graph of $y=x^2$ and label points along the curve by the absolute value of their x-coordinates. (So each point is labeled its horizontal distance from the vertical axis.)



Labeling the points with integer x-coordinates usually suffices. (So in this picture, the blue 3, for example, is the point $(3,9)$ and the red 3 is the point $(-3,9)$.)

Also mark the vertical y-axis with a unit scale. (**Comment:** The graph drawn here is very much not to scale!)

To compute a product of two positive real numbers a and b do the following:

Find the point to the left with red label a, the point to the right with blue label b, and lay a ruler over those two points. Read the location at which the ruler crosses the vertical axis. This location is the product $a\times b$.

Exercise: Really do construct an accurate graph of $y=x^2$ and label the red and blue integer points on it. Use a ruler to estimate the product 2.4×3.9. Also, find an approximate value for $\sqrt{10}$.

WHY THIS WORKS

Explaining why parabolic multiplication works is a lovely challenge to give to algebra students. Here's the explanation:

In allegedly computing the product ab, the method seeks the y-intercept of the line connecting the points $\left(-a,a^2\right)$ and $\left(b,b^2\right)$. This line has slope:

$$\frac{b^2-a^2}{b-(-a)}=b-a$$

and so has equation:

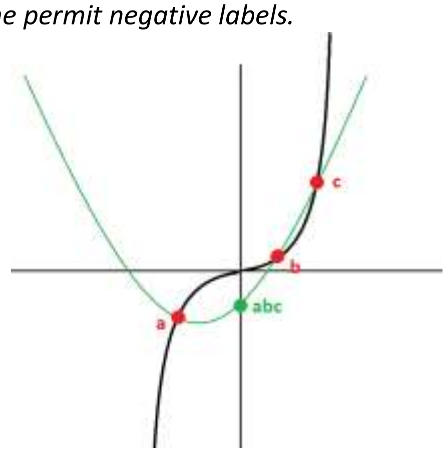
$$y-b^2=(b-a)(x-b).$$

This can be rewritten $y=(b-a)x+ab$, revealing that, indeed, the y-intercept of the line occurs at the product ab.

A THEORETICAL GENERALIZATION:

One can use, in theory, the cubic curve $y=x^3$ to compute the product of any three real numbers a, b, and c.

On the graph of $y=x^3$ label points along the curve by their x-coordinates, but this time permit negative labels.



To find the product of three real numbers a, b, and c simply(!) draw the unique parabola that passes through the three points with those labels. The location at which this parabola crosses the y-axis is the product abc.

The unique parabola that passes through the points $\left(a,a^3\right)$, $\left(b,b^3\right)$, and $\left(c,c^3\right)$:

$$p(x)=a^3\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^3\frac{(x-a)(x-c)}{(b-a)(b-c)}+c^3\frac{(x-a)(x-b)}{(c-a)(c-b)}$$

(Do you see that this formula is quadratic? Do you see that $p(a)=a^3$, $p(b)=b^3$, and $p(c)=c^3$. Check out lesson 5.1 of http://gdaymath.com/courses/quadratics/ to see how to make ridiculously easy sense of this!)

The y-intercept of this quadratic is:

$$p(0)=\frac{a^3bc}{(a-b)(a-c)}+\frac{ab^3c}{(b-a)(b-c)}+\frac{abc^3}{(c-a)(c-b)}$$

which is:

$$\begin{aligned}p(0)&=abc\left(\frac{a^2(b-c)-b^2(a-c)+c^2(a-b)}{(a-b)(a-c)(b-c)}\right)\\&=abc\left(\frac{a^2b-a^2c-ab^2+b^2c+ac^2-bc^2}{a^2b-ab^2+b^2c-a^2c+ac^2-bc^2}\right)\\&=abc.\end{aligned}$$

Indeed this parabola crosses the y-intercept at the value of the product!

> **Question:** What parabola should we draw if we wish to compute a product of the form a^2b (with one term of the product repeated)? What parabola should we draw to compute a number cubed, a^3? (Do we even need a parabola?)

RESEARCH CORNER:
Is there always a lovely relationship connecting the y-intercept of the unique $(n-1)$-degree polynomial that passes through n points on the curve $y=x^n$ and the x-coordinates of those points?

Comment: I do not know who first noticed the clever way to use a quadratic to compute products, nor do I know the origin of the line multiplication method. Further, I am not even sure if I have presented the "correct" versions of the multiplication ideas as they are currently being bandied about in the world of mathematics enthusiasts. Sometimes just half hearing of results or ideas that might or might not exist can be enough to spur on one's personal discovery of cool and interesting mathematics!

Another comment: One should check out James Grimes' Cubic Curve Calculator too: https://www.youtube.com/watch?v=LWkOkM0Gqa4.

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ADDENDUM: Since writing this essay several of my twitter followers pointed out that I missed a swift and elegant solution to problems presented here. Indeed I did!

Consider the cubic curve $y=x^3$ and the parabola, call it $p(x)$, that passes through the three points $\left(a,a^3\right)$, $\left(b,b^3\right)$, and $\left(c,c^3\right)$. Then the polynomial

$$x^3-p(x)$$

is a cubic, with leading coefficient 1, possessing zeros at $x=a$, $x=b$, and $x=c$. So it must factor!

$$x^3-p(x)=(x-a)(x-b)(x-c).$$

Now put $x=0$ to see that $p(0)=abc$.

(Me thinks this now make the research question quite tractable!)