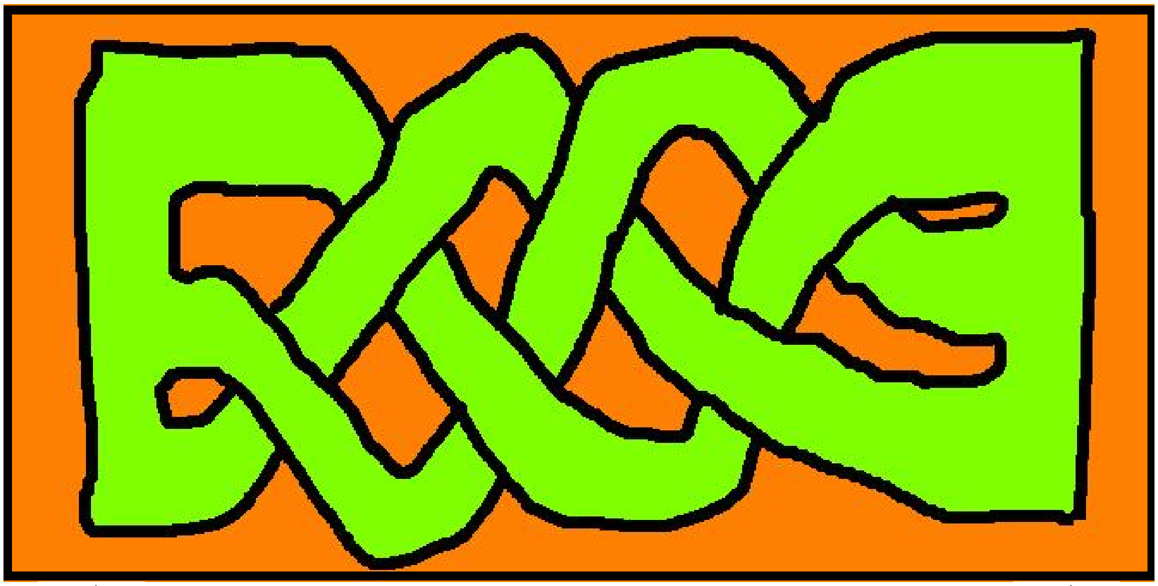


WHAT COOL MATH!

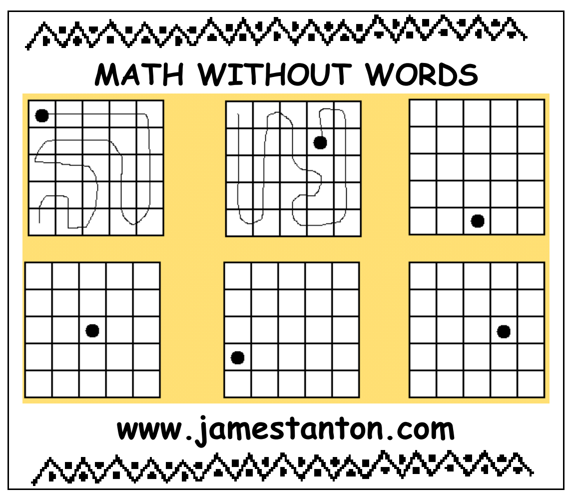
CURIOUS MATHEMATICS FOR FUN AND JOY

The Fifth Math Letter

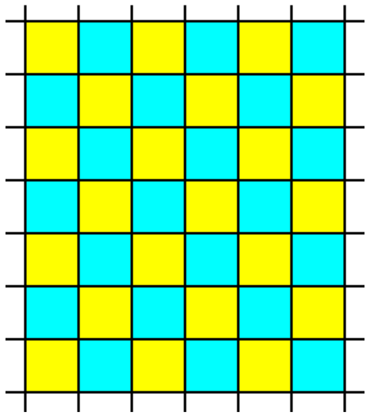
AUGUST 2012

Here is my favourite puzzle of all time. I think it is what made me a mathematician.

(If you are curious to learn how this puzzle affected me so, have a look at the essay with that title on the front page of www.jamestanton.com. One can also obtain there a book of 75 wordless puzzles.)



CHECKERBOARD POWER:
The standard coloring of grid of squares via a checkerboard pattern leads to some profound observations. Here are just a few examples.



1. *Moving only single steps from cell-to-cell in north, east, south or west directions, it is impossible to walk a journey that returns to "start" in an odd number of steps.*

<u>Reason</u>: Each step takes you to a cell of opposite colour. One thus needs an even number of steps to visit any cell of the same starting colour.

2. *Moving freely about a checkerboard a knight will never return to start in an odd number of steps. Nor will a "super-knight" which can move any number of steps in one direction followed by the same number of steps plus one more in an orthogonal direction.*

3. *Each of 25 people standing in a* 5×5 *grid of squares, one person per cell, will never be able to rearrange themselves to end up again one person per cell with each person shifting just one cell over in a north, east, south or west direction.*

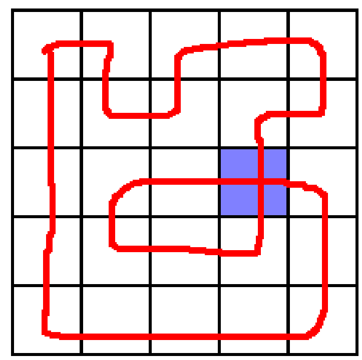
Worse … Jenny, who is one cell away from the center square, will never be able to cheat standing still and have the 24 remaining people each move one cell around her.

<u>Reason</u>: Each person must move one cell over to a cell of opposite colour. In a $5x5$ grid there are 13 people in yellow cells but only 12 green cells for them to step into! And Jenny is in the wrong coloured cell to help out with the matter.

4. *It is impossible to draw a loop in a* 5×5 *grid of squares moving vertically and horizontally and visiting each cell precisely once.*

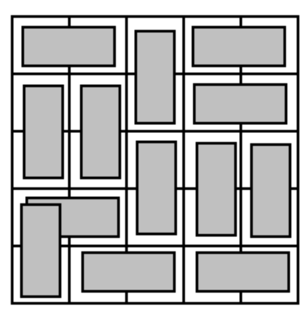
<u>Reason:</u> As we seen, no loop will consist of 25 steps, an odd number.

But it is possible to cheat and accomplish this feat visiting just one cell twice.



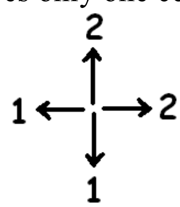
The center cell could never be a "cheat cell." Why? *Nor could the middle cell of left most column be a cheat cell.* Why?

5. *A domino is a* 1×2 *tile that covers two cells of the grid of squares. If we cover a* 5×5 *with 13 dominos, there will be overlap of dominos on one cell.*

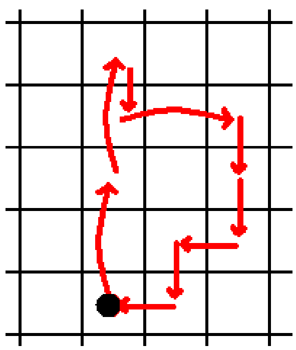


Could the overlap ever occur at a corner cell?

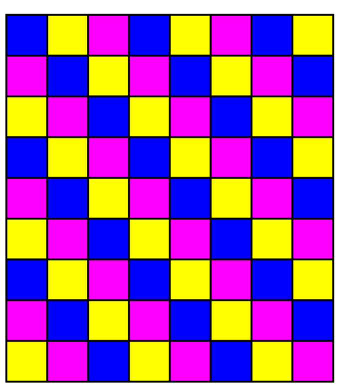
CHECKERBOARD MOTION: Lopsided Charlie moves about the cells of a grid, but in a lopsided manner. When he moves northward, he moves over two squares. He does the same when he moves eastward. When moving southward or westward, on the other hand, he moves only one cell over.



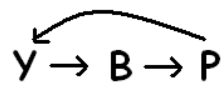
This time it is possible for Charlie to return to start in an odd number of steps. For example, here is a loop with nine steps.



Playing with this for a while, one begins to suspect: *Any loop lopsided Charlie makes consists of a multiple of three steps.* This is indeed true. Here is a "Proof Without Words."

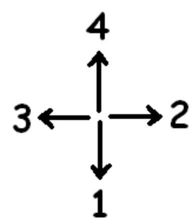


Alright .. Some words! This colouring scheme is designed so that whenever Charlie is on a yellow cell he is sure to move to a blue one next. (Check this!) And whenever he is on a blue cell he is sure to land on a purple one next. And whenever he is on a purple cell, a yellow one is sure to follow. (Check these too!) His steps will thus follow a cycle of three colours:

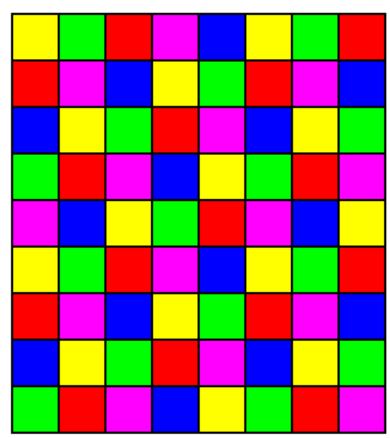


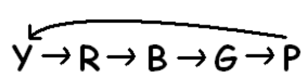
and so a path that comes back to start must consist of a multiple of three count of steps.

Now meet Lulu. She follows motion in the square grid as given by the key:

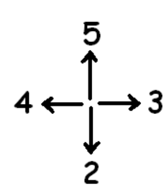


Here is a "proof without words" that any loop she walks must contain a multiple of five steps. Do you see why it works?



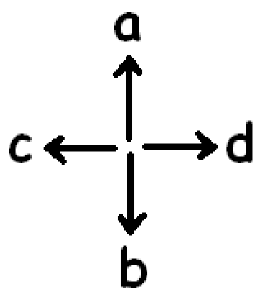


EXERCISE: Find a colouring design that fits the scheme of motion:



BIG CHALLENGE:
These exercises suggest a big question.

Suppose we are given a grid-motion scheme with four numbers a, b, c and d.



What must be true about these numbers for there to exist a colouring pattern of the grid that "fits" the motion? (Is there a set of numbers that has no matching grid colouring?)

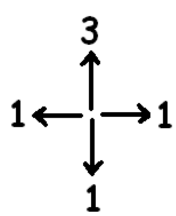
To be clear, by "fit" we mean:

Is it possible to colour each cell of a square grid one of N colours (you get to choose how many!) so that moving either a steps northward, or b steps southward, or c steps westward, or d steps eastward is sure to land you from one colour to the next in a consistent cyclic fashion?

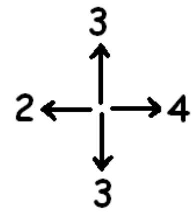
SOME HINTS or CONFUSIONS:

1. All our examples thus far have the property that $a+b=c+d$. Must this always be true?

In what way does the standard checkerboard colouring help or not help with this 3-1-1-1 scheme?



2. So far all our examples have the pair a and b sharing no common factors, as with the pair c and d. Must this be the case? Is it meaningful to consider motion of the following type, for instance?



RESEARCH CORNER:
Half the challenge of this question is deciding what the "meaningful" parameters of the problem are. For example, if $a+b$ and $c+d$ have a common value N, and the two pairs (a,b) and (c,d) each have no common factors, then you might well be able to prove that a suitable colouring scheme with N colours is sure to exist.
But can these restrictions be loosened in any way? The 3-1-1-1 example is curious.

And then we can explore motion in three-dimensional cubical lattices! And how about motion in triangular grids? Can any interesting questions be asked – and answered - in non-square lattices?

Have fun!

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