Curriculum Burst 3: Elite Players and Logarithms

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At a competition with N players, the number of players given elite status is equal to

$$2^{1+\lfloor \log_2(N-1) \rfloor} - N.$$

Suppose that 19 players are given elite status. What is the sum of the two smallest possible values of N? Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

SOURCE: This is question # 19 from the 2011 MAA AMC 12a Competition.

QUICK STATS:

MAA AMC GRADE LEVEL

This question is appropriate for the 12th grade level.

MATHEMATICAL TOPICS

Exponential Functions; Logarithms

COMMON CORE STATE STANDARDS

F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

MATHEMATICAL PRACTICE STANDARDS

MP1 Make sense of problems and persevere in solving them.
MP2 Reason abstractly and quantitatively.
MP3 Construct viable arguments and critique the reasoning of others.
MP7 Look for and make use of structure

PROBLEM SOLVING STRATEGY

ESSAY 4: **ENGAGE IN WISHFUL THINKING**

THE PROBLEM-SOLVING PROCESS:

As always…

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STEP 1: Read the question, have an
emotional reaction to it, take a deep
breath, and then reread the question.
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This problem looks scary and my desire is to just skip over it! (And one does wonder why on Earth anyone would use a formula like that for ascertaining the elite status count!)

Since my emotion here is one of fear, let's try a strategy that often works well in the face of nerves.

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ENGAGE IN WISHFUL THINKING
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I am wary of the formula $2^{1+\lfloor \log_2(N-1) \rfloor} - N$. I wish it were simpler!

Which part is the scariest? The exponent $1+\lfloor \log_2(N-1) \rfloor$. Even though the question probably wants me to do something with it, I am going to pretend it isn't there and just call it M !

The question says the number of elite player is given by

$$2^M - N$$

and this number is 19.

Okay. That's not scary. We have $2^M - N = 19$ and so:

$$N = 2^M - 19.$$

This says that the number of players N is 19 less than a power of two. No worries!

$$N = 32 - 19 = 13$$

or

$$N = 64 - 19 = 45$$

or

$$N = 128 - 19 = 109$$

or

$$N = 256 - 237$$

or ..

Well, there it is! The two smallest numbers N can be are 13 and 45.

Ooh! HANG ON! There are at least 19 players. (There are 19 elite players at least!) The two smallest possible values of N are thus 45 and 109, summing to 154!

EXTENSION:
We completely obviated the issue of understanding, yet alone, working with the expression $2^{1+\lfloor \log_2(N-1) \rfloor}$. Curiously, quantities like these do appear in interesting challenges.

***A Technical Question*:**
Every number N does lie between a pair of consecutive powers of two: $2^k \le N < 2^{k+1}$.

Show that $k = \lfloor \log_2(N) \rfloor$.

What then is the meaning of $2^{\lfloor \log_2(N) \rfloor}$?

A Fun Question:
It is possible to slice a cube of cheese into 27 small cubes in just six planar cuts. (Do you see how?)

If one can stack pieces of cheese between cuts, then it is also possible to slice a cube of cheese into 64 small cubes in just six planar cuts. (Do you see how?)

What is the minimal number of planar cuts needed to slice a cube of cheese into N^3 small cubes? (Assume one can restack pieces of cheese between cuts.)

What is the minimal number of planar cuts needed to slice an $a \times b \times c$ block of cheese into abc small cubes? (The numbers a, b and c are integers.)

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